MTH 310 HW 5 Solutions

Jan 29, 2016

Section 3.3, Problem 9

Prove that if $f : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, it is the identity map.
Answer. Since all isomorphisms are homomorphisms, f maps 0 to 0 by Theorem 3.10. Let $x = f(1) \in \mathbb{Z}$. Then $x = f(1) = f(1*1) = f(1)f(1) = x^2$. The equation $x = x^2$ implies either $x = 0$ or $x = 1$. However, the former case is impossible since f is injective and 0 already maps to 0. Therefore $f(1) = 1$. By Theorem 3.10 again, $f(-1) = -f(1) = -1$.

Finally, let $n \in \mathbb{Z}$. We want to show f(n) = n. If $n = 0$ we have $f(n) = 0$ by 3.10. If $n > 0$ we have $f(n) = f(1 + 1 + ... + 1) = f(1) + ... + f(1) = 1 + ... + 1 = n$ where each ellipses represents addition n times. Finally, if $n < 0, f(n) = f(-(-n)) = -f(-n) = -(-n) = n$ by the above case and theorem 3.10, which allows us to "pull out the negative" (more formally, using the homomorphism property with -1). Thus for all integers $n, f(n) = n$, so f is the identity.

Section 3.3, Problem 11a and 11d

Give one reason why $f : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{Q} \to \mathbb{Q}$ with $f(x) = \sqrt{x}$ and $k(\frac{a}{b}) = \frac{b}{a}$ when $a \neq 0$ and $k(0) = 0$ are not homomorphisms.
Answer. Note that f is not surjective, since there is no $x \in \mathbb{R}$ with $2^x = -1$. Additionally, note that $k(\frac{1}{2} + \frac{1}{1}) = k(\frac{3}{2}) = \frac{2}{3} \neq 3 = 2 + 1 = k(\frac{1}{2}) + k(\frac{1}{1})$.

Section 3.3, Problem 13a

Prove if R, S are rings and $f : RxS \to R$ with $f(r, s) = r$ is not an injective homomorphism if S isn't the zero ring.

Answer. Note that if f was injective, then since $f(0_R, 0_S) = 0_R = f(0_R, 1_S)$ (by theorem 3.10) so $0_S = 1_S$ which implies that S is the zero ring. (Note: If $1 = 0$ in a ring R with identity, then R is the zero ring. To show this, note that for any $x \in R, x = 1x = 0x = 0$. Furthermore, if you don't believe that $0x = 0$ for every $x \in R$, note that $0x = (0+0)x = 0x + 0x$ so by the cancellation law $0x = 0$).